

ON THE SUM $\sum_{d|2^n-1} d^{-1}$

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To the memory of my friend, colleague and collaborator, Eri Jabotinsky

ABSTRACT

Let $\sigma(n)$ be the sum of divisors of n . In this paper we prove $\sigma(2^n - 1) < c(2^n - 1) \log \log n$.

Denote by $\sigma(n)$ the sum of divisors of n . Clearly

$$\sigma(n)/n = \sum_{d|n} \frac{1}{d}.$$

A well known result in number theory states that

$$(1) \quad \limsup_{n \rightarrow \infty} \sigma(n)/n \log \log n = e^\gamma$$

where γ is Euler's constant. In the present note we prove the following

THEOREM.

$$(2) \quad \frac{\sigma(2^n-1)}{2^n-1} = \sum_{d|2^n-1} \frac{1}{d} < c_1 \log \log n.$$

Throughout this paper c_1, c_2, \dots denote positive absolute constants. The theorem is perhaps somewhat surprising since in view of (1) one might have expected that $\sum_{d|2^n-1} 1/d$ can occasionally become as large as $\log n$.

First of all observe that apart from the value of c_1 our Theorem is best possible. To see this let n_k be the product of the first k odd primes and let u_k ($u_k \leq \phi(n_k)$) be the smallest integer with $2^{u_k} \equiv 1 \pmod{n_k}$.

We evidently have by well known results in number theory (prime number theorem and the theorem of Mertens, the p_i 's run through the first k odd primes)

$$\sum_{d|2^{u_k}-1} \frac{1}{d} \geq \prod \left(1 + \frac{1}{p_i}\right) > c_2 \log \log n_k > c_2 \log \log u_k.$$

Received January 22, 1970.

Before we prove our Theorem we state a few problems and results. Put

$$\varepsilon_n = \sum \frac{1}{d}, d|2^n - 1, d \nmid 2^m - 1 \text{ for } m < n.$$

A well known result of Romanoff [1] states that $\sum_{n=1}^{\infty} \varepsilon_n/n$ converges. This follows easily from $\sum_{k=1}^n \varepsilon_k < c_3 \log n$. Probably

$$\sum_{k=1}^n \varepsilon_k = (c_4 + o(1)) \log n$$

and

$$c_5 < \sum_{k=n}^{2n} \varepsilon_k < c_6.$$

It seems likely that

$$\limsup_{n \rightarrow \infty} n \varepsilon_n = \infty, \liminf_{n \rightarrow \infty} n \varepsilon_n = 0$$

and that $n \varepsilon_n$ has a distribution function. I can prove only that $\varepsilon_n \rightarrow 0$ and in fact I can even prove that

$$(3) \quad \sum_{\substack{d|2^n-1 \\ d > n}} \frac{1}{d} \rightarrow 0.$$

Very likely

$$\varepsilon_n = O\left(\frac{1}{n^{1-\delta}}\right)$$

for every $\delta > 0$ but I could not even prove $\varepsilon_n < 1/n^{c_7}$. I could not obtain a satisfactory estimation of the sum (3). I proved that

$$\sum_{d|2^n-1} \frac{1}{d} = \frac{\sigma(2^n-1)}{2^n-1}$$

has a distribution function, but we do not discuss the proof here.

Now we prove our theorem. To prove (2) it will suffice to show that (p prime)

$$(4) \quad \sum_{p|2^n-1} \frac{1}{p} < \log \log \log n + c_8.$$

(4) implies (2) by $e^x > 1 + x$.

To prove (4) write

$$(5) \quad \sum_{p|2^n-1} \frac{1}{p} = \sum_{d|n} \sum_d \frac{1}{p} = \Sigma_1 + \Sigma_2 + \Sigma_3$$

where in Σ_d p runs through the primes p satisfying $p \mid 2^d - 1$, $p \nmid 2^{d'} - 1$, $d' < d$, $d' \mid n$, and in Σ_1 $d \leq (\log n)^{16}$, in Σ_2 $d > (\log n)^{16}$, $p < n$ and in Σ_3 $p \geq n$. First we estimate Σ_1 . Clearly $2^d - 1$ has fewer than d prime factors, hence Σ_1 has fewer than $(\log n)^{32}$ summands. Thus Σ_1 is less than the sum of the reciprocals of the first $[(\log n)^{32}]$ primes. Hence from the prime number theorem (or a more elementary theorem)

$$(6) \quad \Sigma_1 < \sum_{p < (\log n)^{32}} \frac{1}{p} < \log \log \log n + c_9.$$

Next we estimate Σ_2 , this will be considerably more difficult than the estimation of Σ_1 . First of all put

$$(7) \quad \Sigma_2 = \Sigma'_2 + \Sigma''_2,$$

where in

$$(8) \quad \Sigma'_2 = \sum_{d > (\log n)^{16}} \Sigma_{2,d} \frac{1}{p}$$

only primes $p > d^3$ occur in the inner sum and in Σ''_2 are the primes $d < p < d^3$ $p \mid n$, $p \equiv 1 \pmod{d}$ (all prime factors of $2^d - 1$ which do not divide any $2^{d'} - 1$, $d' \mid d$ are well known to be $\equiv 1 \pmod{d}$)).

$2^d - 1$ has fewer than d prime factors, thus

$$(9) \quad \Sigma_{2,d} \frac{1}{p} < \frac{1}{d^2}.$$

From (8) and (9) we have

$$(10) \quad \Sigma'_2 = \sum_{d > (\log n)^{16}}^{\infty} \frac{1}{d^2} = o(1).$$

Hence we only have to estimate Σ''_2 and this will be the only difficult part of our note. Denote by $q_1 < q_2 < \dots < q_s < n$ the sequence of primes which occur in Σ'' . In other words for every q_i there is a d satisfying

$$(11) \quad d > (\log n)^{16}, q_i \equiv 1 \pmod{d}, d^3 > q_i, q_i \mid 2^d - 1, q_i \nmid 2^{d_1} - 1 \text{ for } d_1 < d, d_1 \mid n,$$

(since $(2^a - 1, 2^b - 1) = 2^{(a,b)} - 1$, $d_1 \mid n$ could be replaced by $d_1 \mid d$).

Thus

$$(12) \quad \Sigma'' = \sum_i \frac{1}{q_i} = \sum_{k=4}^{\infty} \Sigma_k \frac{1}{q_i}$$

where in Σ_k

$$(13) \quad (\log n)^{2k} < q_i \leq (\log n)^{2k+1}.$$

Next we estimate $\Sigma_k 1/q_i$. If q_i occurs in Σ_k we have by (11) that there is a $d \mid n$ for which $q_i \equiv 1 \pmod{d}$, $d^3 > q_i$, or by (13)

$$(14) \quad (q_i - 1, n) > (\log n)^{2k-2}.$$

Let $(\log n)^{2k} < x < (\log n)^{2k+1}$ ($k \geq 4$, $x \leq n$). Denote by $Q(x)$ the number of primes $q < x$ which satisfy (14). Let r_1, \dots , be the prime factors of n . To estimate $Q(x)$ from above, we first estimate from above (p runs through all the primes $\leq x$)

$$(15) \quad A(n, x) = \prod_{p < x} (p - 1, n).$$

We evidently have

$$(16) \quad A(n, x) \leq \prod_{r_i \mid n} \prod_{l=1}^{\infty} r_i^{\pi(x, r_i^l, 1)} = \Pi_1 \Pi_2$$

where $\pi(x, d, 1)$ denotes the number of primes $p \leq x$ satisfying $p \equiv 1 \pmod{d}$ and in Π_1 , $r_i^l \leq (\log n)^{10}$ and in Π_2 , $r_i^l > (\log n)^{10}$.

By a theorem of Brun-Titchmarsh [2] we have for $q_i^l < (\log n)^{10}$, $x > (\log n)^{16}$

$$(17) \quad \pi(x, q_i^l, 1) < c_{10} \frac{x}{q_i^l \log x}.$$

From (17) we obtain by the theorem of Mertens, ($\exp z = e^z$)

$$(18) \quad \begin{aligned} \Pi_1 &\leq \prod_{r_i^l < (\log n)^{10}} r_i^{c_{10} x / r_i^l \log x} \leq \exp \frac{c_{10} x}{\log x} \sum_{r_i \leq (\log n)^{10}} \sum_{l=1}^{\infty} \frac{\log r_i}{r_i^l} \\ &\leq \exp \frac{c_{11} x \log \log n}{\log x}. \end{aligned}$$

Next we estimate Π_2 . If $r_i^l > (\log n)^{10}$ we use the trivial estimate

$$(19) \quad \pi(x, r_i^l, 1) < \frac{x}{r_i^l} < \frac{x}{(\log n)^{10}}.$$

The number of prime factors of n (multiple factors counted multiply) is clearly at most $\log n / \log 2$, thus from (19) ($x \leq n$)

$$(20) \quad \Pi_2 < \prod_{r_i \mid n} r_i^{x / (\log n)^{10}} < x^{2x / (\log n)^9} < x^{2x / (\log x)^9} = \exp \frac{2x}{(\log x)^8}.$$

From (16), (18) and (20) we have

$$(21) \quad A(n, x) < \exp c_{12} (x \log \log n / \log x).$$

From (20) and the definition of $Q(x)$ we have

$$(22) \quad A(n, x) > (\log n)^{2^{k-2}Q(x)}.$$

Thus finally from (21) and (22)

$$(23) \quad Q(x) < \frac{c_{12}x}{2^{k-2}\log x}.$$

From (23) we immediately obtain

$$(24) \quad \sum_k \frac{1}{q_i} < c_{13}/2^k$$

and thus from (23)

$$(25) \quad \sum_{k=4}^{\infty} \sum_k \frac{1}{q_i} < c_{13}.$$

From (6), (9), (11) and (24) we finally have

$$(26) \quad \Sigma_2 < c_{13} + o(1).$$

The estimation of Σ_3 is very simple. $2^n - 1$ clearly has fewer than n prime factors, thus

$$(27) \quad \Sigma_3 < 1$$

(6), (26) and (27) proves (3) which completes the proof of our Theorem.

Perhaps the following stronger result holds:

Let $3, 5 \dots p_k \leq n < 3.5 \dots p_{k+1}$. Then $(p_i$ runs through the consecutive odd primes)

$$(28) \quad \max_{m \leq n} \sum_{p|2^m-1} \frac{1}{p} = \sum_{i=1}^k \frac{1}{p_i} + o(1),$$

but the methods used in this note are not strong enough to decide (28).

Clearly our proof gives that for every a

$$\sum_{d|a^n-1} \frac{1}{d} < c_a \log \log n,$$

but I cannot decide whether

$$\sum_{d|2^n-3} \frac{1}{d} < c_{14} \log \log n$$

holds.

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